§ Lines & Planes

Lines

Let $\vec{v} = (v_1, v_2, v_3)$ be a vector, $P = (P_1\ P_2\ P_3)$ be a point, then we can define the line passing through P with dirction $\vec{v}$ to be

$$L = P + \{t \cdot \vec{v} \mid t \in \mathbb{R}\}$$

$$= \{(v_1 t + P_1,\ v_2 t + P_2,\ v_3 t + P_3)\}$$ See Pic. 1

We call t a parameter of L.

If we take another point $Q = (q_1, q_2, q_3) \in L$ $\vec{w} \,/\!/\, \vec{v}$, then

$$L = Q + \{t\vec{w} \mid t \in \mathbb{R}\}$$
$$= (w_1 t + q_1,\ w_2 t + q_2,\ w_3 t + q_3)$$ See Pic. 2

So this expression is not unique. Each of them is called a parametrization.

Example: $P = (1, 0, 1)$, $\vec{v} = (3, 7, -2)$

$$L = \{(3t+1,\ 7t,\ -2t+1)\}$$

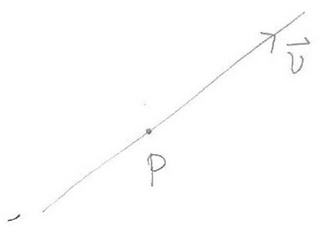
Take $Q = (4, 7, -1)$ $(t=1)$, then we can also write

$$L = \{(3t+4,\ 7t+7,\ -2t-1)\}$$

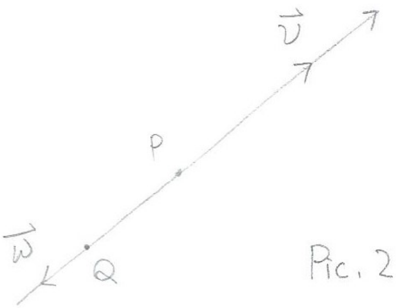
We can also choose a nonlinear parameter:

$$L = \{(3s^3+4,\ 7s^3+7,\ -2s^3-1)\}$$

is also a parametrization (change of variable, $t = s^3$)



Pic. 1



Pic. 2

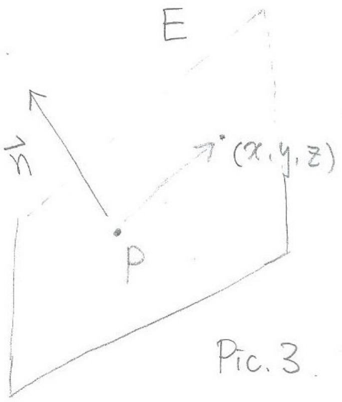
Planes:

Using Dot product: Let $P = (P_1\ P_2\ P_3)$ be a point in $\mathbb{R}^3$,

$\vec{n} = (n_1\ n_2\ n_3)$ be a vector, then

we can define
$$E = \{(x, y, z) \mid ((x, y, z) - P) \cdot \vec{n} = 0\}$$

see Pic. 3



Pic. 3

So for any point $(x, y, z) \in E$, it must satisfies the equation

$$n_1 x + n_1 P_1 + n_2 y + n_2 P_2 + n_3 z + n_3 P_3 = 0$$

We can simplify it as

$$n_1 x + n_2 y + n_3 z + m = 0$$

where $n_1\ n_2\ n_3,\ m \in \mathbb{R}$. This is the equation of a plane.

Properties: • If two equations determine the same plane, then they are the same up to a scalar.

Example: $\{2x + 3y - z + 7 = 0\} = E$
$$= \{4x + 6y - 2z + 14 = 0\}$$

• The coefficients (n_1, n_2, n_3) will be a normal vector.

Applications: Directions and Angles

If $E_1\ E_2$ are two planes with

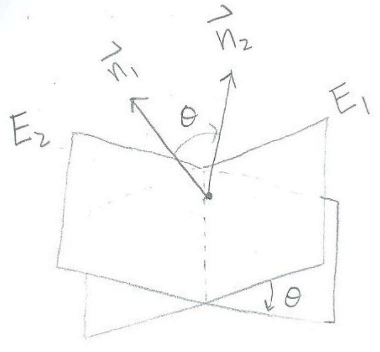
$$E_1 = \{a_1 x + b_1 y + c_1 z + d_1 = 0\}$$
$$E_2 = \{a_2 x + b_2 y + c_2 z + d_2 = 0\}$$

$E_1 \parallel E_2$ iff $(a_1, b_1, c_1) = r(a_2, b_2\ c_2)$ for some $r \neq 0$

and $d_1 \neq r d_2$

(otherwise $E_1 = E_2$).

The angle between 2 planes: See. Pic. 4



Pic. 4

"The angle between E_1 and E_2"

= "the angle between $\vec{n}_1$ and $\vec{n}_2$"

So $\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

$$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)$$

Applications: Intersections

- Let $L = P + \{ t\vec{v} \}$

  $E = \{ ax + by + cz + d = 0 \}$

To find the intersection between L and E, we just plug in $(v_1 t + P_1,\ v_2 t + P_2,\ v_3 t + P_3)$ into

$$ax + by + cz + d = 0$$

Example:

Let $L = (3t+1,\ 7t,\ -2t+1)$

$E = \{2x + 3y - z + 7 = 0\}$

Solve: $2(3t+1) + 3(7t) - (-2t+1) + 7 = 0$

$\Rightarrow 6t + 2 + 21t + 4t - 1 + 7 = 0$

$\Rightarrow 31t + 8 = 0 \Rightarrow t = -\frac{8}{31}$, So $L \cap E = \{ (\frac{7}{31}, -\frac{56}{31}, \frac{47}{31}) \}$

- Let $E_1 = \{ a_1 x + b_1 y + c_1 z + d_1 = 0 \}$

$E_2 = \{ a_2 x + b_2 y + c_2 z + d_2 = 0 \}$ and $E_1 \nparallel E_2$.

Then $E_1 \cap E_2 = L$ for some line L.

To determint L, we need a point P and a vector $\vec{V}$

$P = (P_1\ P_2\ P_3)$ can be any point on $E_1 \cap E_2$.

Since $E_1 \nparallel E_2 \Rightarrow \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \det \begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix}, \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$

are not all zero.

Suppose $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$. By taking $z = 0$, we have

$$\begin{cases} a_1 x + b_1 y + d_1 = 0 \\ a_2 x + b_2 y + d_2 = 0 \end{cases}$$

Then we can solve this system to get $x = P_1$, $y = P_2$.

$P = (P_1\ P_2\ 0) \in E_1 \cap E_2$

To find $\vec{V}$: See Pic 4

$\vec{V} = \vec{n_1} \times \vec{n_2}$ where $\vec{n_1} = (a_1\ b_1\ c_1)$

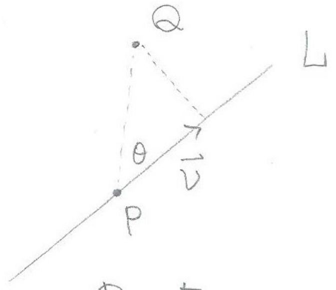
$\vec{n_2} = (a_2\ b_2\ c_2)$.

Applications: Distance

A point to a line:

$$d(Q, L) = |PQ| \sin\theta \quad \text{see Pic. 5}$$

$$= \frac{|\vec{PQ} \times \vec{v}|}{|\vec{v}|}.$$



Pic. 5

A point to a Plane:

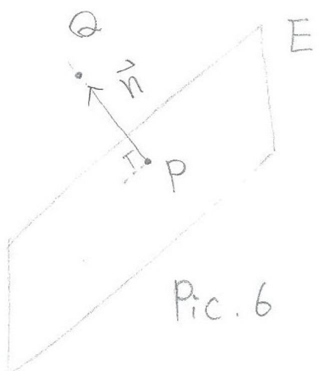
$$d(Q, E) = d(P, Q) \quad \text{See Pic. 6}$$

So we can solve

$$P \in E \cap L$$

where $L = Q + \{t\vec{n}\}$.

Then compute $d(P, Q) = |\vec{PQ}|$



Pic. 6

§ Quadric surfaces :

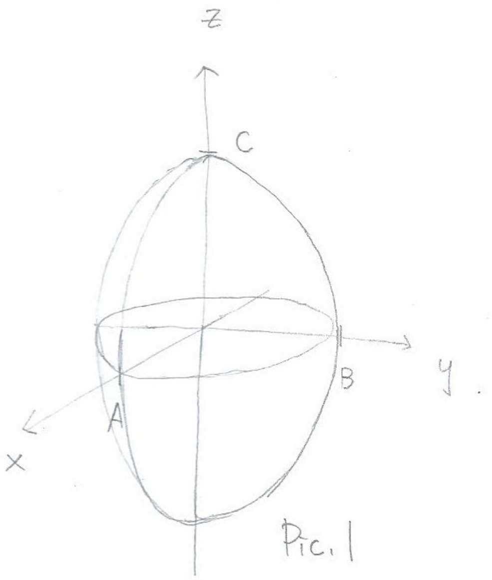
A quadric surface is a surface satisfying eq. of the form

$$Q = \{ ax^2 + by^2 + cz^2 + dx + ey + fz + g = 0 \}$$

(general case will be discussed later)

3 standard types :

ellipsoid : $\dfrac{x^2}{A^2} + \dfrac{y^2}{B^2} + \dfrac{z^2}{C^2} = 1$ See Pic. 1



Pic. 1

Hyperboloid (one sheet) :

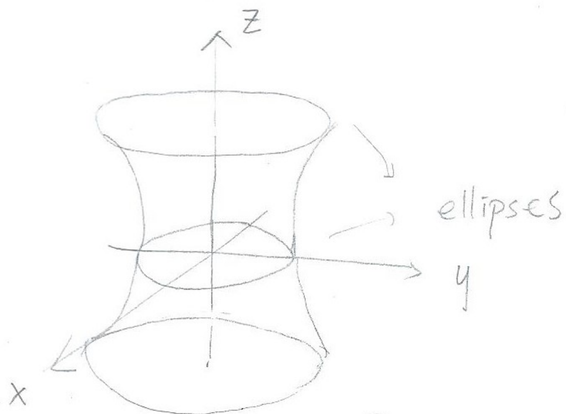
$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{z^2}{C^2} = 1$$

See. Pic 2.

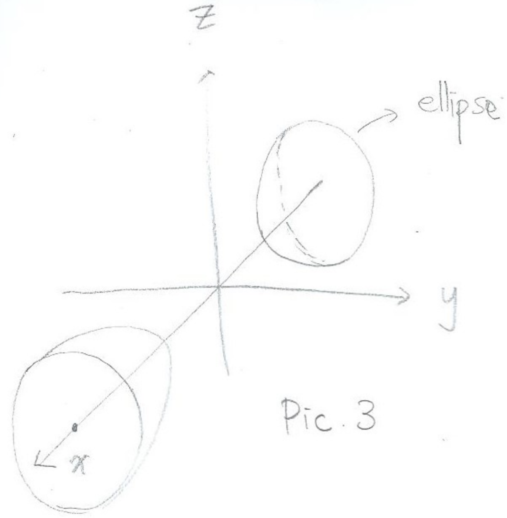
Hyperboloid (2-sheets)

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} - \frac{z^2}{C^2} = 1$$

See Pic. 3



Pic. 2



Pic. 3

Elliptical Cone:

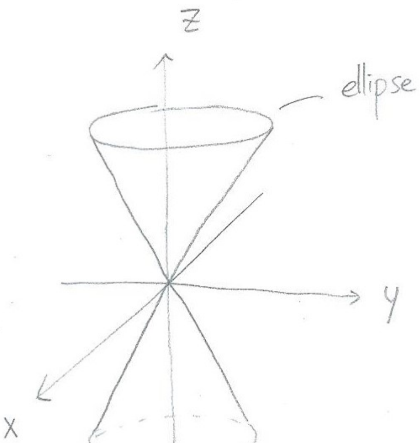
$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} - \frac{Z^2}{C^2} = 0$$

See. Pic. 4

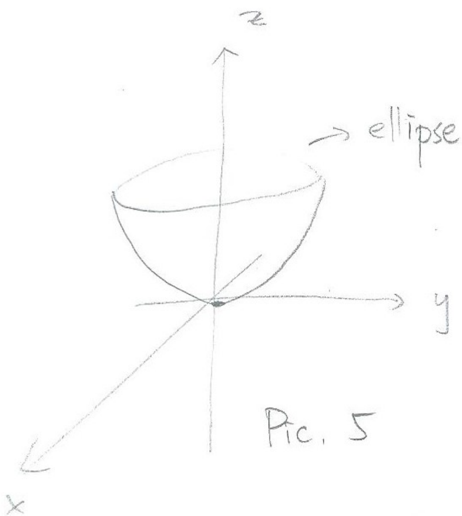
Degenerate types:

Elliptical Paraboloid: $\frac{X^2}{A^2} + \frac{y^2}{B^2} = \frac{Z}{C}$ Pic. 5

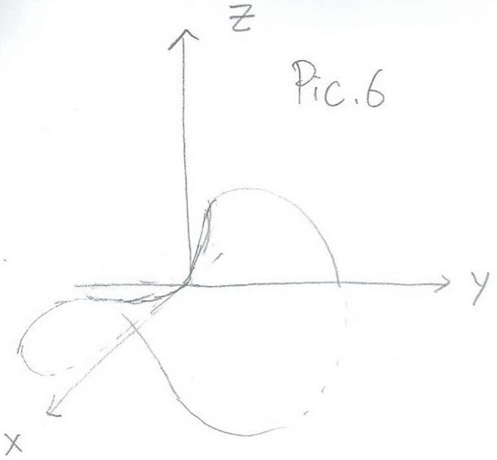
Hyperbolic Paraboloid: $\frac{X^2}{A^2} - \frac{y^2}{B^2} = \frac{Z}{C}$ Pic. 6



Pic. 4



Pic. 5



Pic.6

Simplification: $ax^2 + by^2 + cz^2 + dx + ey + fz + g = 0$

If $a \neq 0$ $\quad ax^2 + dx = a\left(x^2 + \frac{d}{a}x\right)$

$$= a\underbrace{\left(x + \frac{d}{2a}\right)}_{\hat{x}}{}^2 \underbrace{-\left(\frac{d^2}{4a}\right)}_{\hat{d}}$$

$$ax^2 + dx = a\hat{x}^2 + \hat{d}$$

So by changing the variable
$$\hat{x} = x + \frac{d}{2a} \quad \text{if } a \neq 0$$
$$\hat{y} = y + \frac{e}{2b} \quad \text{if } b \neq 0$$
$$\hat{z} = z + \frac{f}{2c} \quad \text{if } c \neq 0$$

We have

$$a\hat{x}^2 + b\hat{y}^2 + c\hat{z}^2 + \text{Some constant} = 0$$

$\Rightarrow$ Q will be one of the form above

or a Plane.

General Cases: a surface satisfying the following eq is also called a Quadric surface:

$$Q = \{ ax^2 + by^2 + cz^2 + pxy + qyz + rzx + dx + ey + fz + g = 0 \}$$

In this case, we need to change our coordinate

$$ax^2 + by^2 + cz^2 + pxy + qyz + rzx$$

$$= (x\ y\ z) \underbrace{\begin{pmatrix} a & \frac{1}{2}p & \frac{1}{2}r \\ \frac{1}{2}p & b & \frac{1}{2}q \\ \frac{1}{2}r & \frac{1}{2}q & c \end{pmatrix}}_{\substack{\shortparallel \\ M}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Find a matrix R s.t.

$$RMR^{-1} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad \text{for some } \lambda_1\ \lambda_2\ \lambda_3$$

Taking $R \begin{pmatrix} x \\ y \\ z \end{pmatrix} := \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ as our new coordinate

$$\Rightarrow Q = \{ \lambda_1 u^2 + \lambda_2 v^2 + \lambda_3 w^2 + \text{some linear term} \}$$